

CORRELATIONS BETWEEN ZEROS AND CRITICAL POINTS OF RANDOM ANALYTIC FUNCTIONS

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ABSTRACT. We study the two-point correlation $K_n^m(z, w)$ between zeros and critical points of Gaussian random holomorphic sections s_n over Kähler manifolds. The critical points are points $\nabla_{h^n} s_n = 0$ where ∇_{h^n} is the smooth Chern connection with respect to the Hermitian metric h^n on line bundle L^n . The main result is that the rescaling limit of $K_n^m(z_0 + \frac{u}{\sqrt{n}}, z_0 + \frac{v}{\sqrt{n}})$ for any $z_0 \in M$ is universal as n tends to infinity. In fact, the universal rescaling limit is the two-point correlation between zeros and critical points of Gaussian analytic functions for the Bargmann-Fock space of level 1. Furthermore, there is a 'repulsion' between zeros and critical points for the short range; and a 'neutrality' for the long range.

1. INTRODUCTION

In this article, we study the two-point correlation between critical points and zeros of random analytic functions and its generalization to random holomorphic sections on Kähler manifolds. The famous Gauss-Lucas Theorem states that the holomorphic critical points of any polynomial of complex one variable are contained in the convex hull of its zeros. This implies that some non-trivial correlations between zeros and critical points of random polynomials must exist. It seems that the analogous properties should exist for random holomorphic sections on Kähler manifolds. In [6], the author studied two conditional expectations on Riemann surfaces: the expected density of zeros of Gaussian random sections with a conditioning critical point and the expected density of critical points with a fixed zero. It's proved that both conditional densities have universal rescaling limits but the short range behaviors are quite different: there is a 'neutrality' between critical points and the conditioning zero while there is a 'repulsion' between zeros and the conditioning critical point. In this paper, we further study the two-point correlation between zeros and critical points of Gaussian random holomorphic sections and its rescaling limit. The essential difference to the Gauss-Lucas setting is that the critical points are defined as zeros of the derivative of the smooth Chern connection ∇_{h^n} with respect to the Hermitian metric h^n on the line bundle L^n instead of the holomorphic derivative $\frac{\partial}{\partial z}$ (a meromorphic connection). Hence, the two-point correlation should depend on the geometry, i.e., metrics defined on line bundles and Kähler manifolds. But we will show that the rescaling limit of the two-point correlation is universal. In fact, the universal rescaling limit is the two-point correlation between zeros and critical points of Gaussian analytic functions for the Bargmann-Fock space of level 1. Such universal rescaling limit phenomenon was first proved by Bleher-Shiffman-Zelditch in [3] for the two-point correlation between zeros of

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Gaussian random holomorphic sections. In this article, we will generalize Bleher-Shiffman-Zelditch's method to derive the universal rescaling limit of the two-point correlation between zeros and critical points on Kähler manifolds. We will show that the rescaling two-point correlation will tend to 0 for the short range and tend to a positive constant (which only depends on the dimension) for the long range. Roughly speaking, there is a 'repulsion' between zeros and critical points for the short range and a 'neutrality' for the long range which means that zeros and critical points behave independently if they are far apart.

1.1. Main results. To state our results, we need to recall some basic definitions of Gaussian random holomorphic sections of a line bundle (see §2). We let $(L, h) \rightarrow (M, \omega)$ be a positive Hermitian holomorphic line bundle over a compact Kähler manifold of complex m -dimensional. We denote $H^0(M, L^n)$ as the space of global holomorphic sections of the n -th tensor power of L . The Hermitian metric h will induce an inner product on $H^0(M, L^n)$ (10) and thus induces a Gaussian measure $d\gamma_{d_n}$ on $H^0(M, L^n)$, where d_n is the dimension of $H^0(M, L^n)$. A special case is when $M = \mathbb{CP}^1 \cong S^2$ and $L = \mathcal{O}(1)$ the hyperplane line bundle, $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ is the space of homogeneous polynomials of degree n . There is a classical Fubini-Study metric defined on the line bundle $(\mathcal{O}(1), h_{FS}) \rightarrow (\mathbb{CP}^1, \omega_{FS})$ which will induce an inner product on $H^0(\mathbb{CP}^1, \mathcal{O}(n))$. Hence, it will induce a Gaussian measure on the space of homogeneous polynomials of degree n ; the corresponding random polynomials are called Gaussian $SU(2)$ polynomials which are invariant under the rotation on S^2 , or equivalently, the $SU(2)$ action on \mathbb{CP}^1 .

Throughout the article, we assume our line bundle is polarized, i.e., $h = e^{-\phi}$, where ϕ is the smooth local Kähler potential such that $\omega = \partial\bar{\partial}\phi$. Given a global holomorphic section s_n , we write $s_n = f_n e^{\otimes n}$ in a local coordinate patch where f_n is a holomorphic function, the Chern connection of s_n is given by [8]

$$(1) \quad \nabla_{h^n} s_n = \sum_{i=1}^m \left(\frac{\partial f_n}{\partial z_i} - n \frac{\partial \phi}{\partial z_i} f_n \right) e^{\otimes n} \otimes dz_i.$$

The critical points of holomorphic sections are points where $\nabla_{h^n} s_n = 0$. Note that $\nabla_{h^n} s_n = 0$ is only a smooth equation instead a holomorphic equation since ϕ is smooth; furthermore, the solution depends on the geometry. This is the essential difference in our study between complex manifolds and the complex plane. For example, let M be a compact Riemann surface, the total number of zeros of non-zero holomorphic sections of the positive holomorphic line bundle $L^n \rightarrow M$ is $c_1(L)n$ which is topologically invariant [8], but the total number of critical points is not topological. Deterministically, given a holomorphic section, one can not tell how many critical points it has. But on average, it's proved in [4, 5] that the expected number of critical points has the asymptotics,

$$\begin{aligned} & \mathbf{E}(\# \text{ of critical points defined by Chern connection}) \\ &= \frac{5}{3}c_1(L)n + \frac{7}{9}(2g-2) + (\text{non-topological term})n^{-1} + \dots, \end{aligned}$$

where g is the genus of the Riemann surface and the non-topological terms depend on the global geometry where the curvatures of the Kähler metric are involved. Take Gaussian $SU(2)$ polynomials $p_n(z)$ for example, the holomorphic derivative $\frac{\partial p_n}{\partial z} = 0$ always gives $n-1$ critical points; but the average number of critical points defined by the smooth Chern derivative is asymptotic to $\frac{5}{3}n$ (recall $c_1(\mathcal{O}(1)) = 1$).

We define the two-point correlation between zeros and critical point of Gaussian random sections with respect to $(H^0(M, L^n), d\gamma_{d_n})$ as

$$(2) \quad K_n^m(z, w) := \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})} \left(\sum_{z: s_n(z)=0} \delta_z \otimes \sum_{w: \nabla_{h^n} s_n(w)=0} \delta_w \right).$$

Note that given a non-zero global holomorphic section, the zero set is an algebraic variety of codimension 1 and the set of critical points is codimension m . Thus, $K_n^m(z, w)$ is a $(m+1, m+1)$ -current on $M \times M$ in the sense of distribution,

$$(3) \quad \begin{aligned} & \int_{M \times M} \psi(z) \varphi(w) K_n^m(z, w) \\ &= \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})} \left(\int_{\{z: s_n(z)=0\}} \psi(z) \sum_{w: \nabla_{h^n} s_n(w)=0} \varphi(w) \right) \end{aligned}$$

where ψ is any smooth $(m-1, m-1)$ -current on M and φ is a smooth test function.

The purpose of the article is to study the typical spacing between zeros and critical points. We rescale the global expression of $K_n^m(z, w)$ by a factor \sqrt{n} at any fixed point $z_0 \in M$, i.e., we enlarge the local geodesic ball by a factor \sqrt{n} . Note that $K_n^m(z, w)$ is a $(m+1, m+1)$ -current depending on the Hermitian metric h on the line bundle, but our main result claims that its rescaling limit is universal,

Theorem 1. *The rescaling of the $(m+1, m+1)$ -current of the two-point correlation of zeros and critical points of Gaussian random sections with respect to $(H^0(M, L^n), d\gamma_n)$ has the following pointwise universal limit,*

$$(4) \quad \lim_{n \rightarrow \infty} K_n^m(z_0 + \frac{u}{\sqrt{n}}, z_0 + \frac{v}{\sqrt{n}}) = K_{BF}^m(u, v) \frac{d\ell_u}{\pi} \wedge \frac{(d\ell_v)^m}{\pi^m m!},$$

where we denote $d\ell_z := \frac{i}{2} \sum_{j=1}^m dz_j \wedge d\bar{z}_j$ such that $\frac{(d\ell_z)^m}{m!}$ is the Lebesgue measure on \mathbb{C}^m . In fact, $K_{BF}^m(u, v)$ is the two-point correlation function between zeros and critical points of Gaussian analytic functions of the Bargmann-Fock space of level 1; the explicit expression of $K_{BF}^m(u, v)$ is given by (46). Furthermore, $K_{BF}^m(u, v)$ is a function of $|u - v|$ and it admits the following pointwise limits,

$$(5) \quad \lim_{|u-v| \rightarrow 0} K_{BF}^m(u, v) = 0 \quad \text{and} \quad \lim_{|u-v| \rightarrow \infty} K_{BF}^m(u, v) = c_m.$$

where c_m is a constant only depending on the dimension, in particular, $c_1 = \frac{5}{3}$.

To prove this, we will first derive a Kac-Rice type formula on Kähler manifolds. We will see that the two-point correlation can be expressed by the Bergman kernel and its derivatives up to order 4. It's well-known that the Bergman kernel on any Kähler manifold has a universal rescaling limit – the Bergman kernel for the Bargmann-Fock space of level 1. Hence, the two-point correlation will admit a universal rescaling limit; the limit is actually the two-point correlation of Gaussian analytic functions of the Bargmann-Fock space of level 1.

Theorem 1 determines some local behaviors between critical points and zeros. Intuitively, the rescaling limit of the two-point correlation measures the asymptotic probability of finding critical points and zeros in the small geodesic ball of radius of order $n^{-\frac{1}{2}}$. Roughly speaking, let's take the 1-dimensional Riemann surfaces for example, the rescaling limit $K_{BF}^m(u, v)$ tending to 0 as $|u - v| \rightarrow 0$ indicates that it's unlikely to find a zero and a critical point nearby simultaneously, i.e., there is

a ‘repulsion’ between zeros and critical points. The limit $K_{BF}^m(u, v)$ tending to $\frac{5}{3}$ as $|u - v| \rightarrow \infty$ indicates that zeros and critical points can not ‘feel’ each other for the long range, or equivalently, there is no correlation for the long range.

A possible explanation for the ‘repulsion’ phenomenon is as follows. For the positive holomorphic line bundle, it’s well known that the local minima of the h -norm $|s_n|_{h^n}$ are its zeros and the local maxima/saddle points of $|s_n|_{h^n}$ are obtained at the critical points $\nabla_{h^n} s_n = 0$ [8]. Intuitively, at the zero of $|s_n|_{h^n}$, $|s_n|_{h^n}$ is ‘turning up’ and it is very possible that it takes a while for $|s_n|_{h^n}$ to reach the local maxima/saddle, or equivalently, the process can not touch the local maxima/saddle immediately after it leaves 0, which implies that a ‘repulsion’ could occur between local minima and local maxima/saddle of $|s_n|_{h^n}$. This might explain that a ‘repulsion’ exists between zeros and critical points of s_n .

1.2. Comparisons between Meromorphic and Chern connections. As a remark, the two-point correlation between zeros and holomorphic critical points has been studied recently in [9, 10, 11] for Gaussian $SU(2)$ polynomials on the complex plane \mathbb{C} . In fact, the Gaussian $SU(2)$ polynomials can be viewed as meromorphic functions on \mathbb{CP}^1 and the holomorphic derivative $\frac{\partial}{\partial z}$ can be viewed as a meromorphic connection on \mathbb{CP}^1 which has a pole at infinity. In [9], the two-point correlation function between zeros and the holomorphic critical points is derived by the Poincaré-Lelong formula (but the author did not derive the rescaling limit). In [10, 11], it is also proved that zeros and critical points appear in rigid pairs, to be more precise, given a zero, with high probability there is a unique critical point in the ball of radius of order n^{-1} around the zero.

The smooth Chern connection plays an important role in our results compared with meromorphic connections. As we show in this article, the rescaling limit $K_n(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}})$ of the two-point correlation between zeros and critical points (defined by the smooth Chern connection) is universal if we rescale the local domain by a factor $n^{-\frac{1}{2}}$, roughly speaking, this implies that the typical spacing between zeros and critical points is $n^{-\frac{1}{2}}$. Let $K_n^{mero}(z, w)$ be the two-point correlation between zeros and critical points defined by a meromorphic connection $\frac{\partial s_n}{\partial z} = 0$ for Gaussian random holomorphic sections s_n . In [7], we show that $K_n^{mero}(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}})$ also admits a universal limit. The above two rescaling limits exist since the covariance kernel of Gaussian random sections s_n is the Bergman kernel (see (29)) and the Bergman kernel has the universal rescaling limit $e^{z\bar{w}}$ (see §4). In fact, following the main idea in [3] and our proof in §4, in order to derive the rescaling limit, it’s enough to consider the Gaussian analytic function

$$f(z) = \sum_{j=0}^{\infty} \frac{a_j}{\sqrt{j!}} z^j,$$

where a_j are i.i.d. standard complex Gaussian random variables with mean 0 and variance 1. Both the limits $K_n(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}})$ and $K_n^{mero}(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}})$ are universal and obtained by the corresponding ones for $f(z)$. But the behaviors of these two rescaling limits of K_n and K_n^{mero} are quite different. First note that the distribution of zeros of $f(z)$ is invariant under the translation and rotation of the complex plane [12]. Now recall (31) of the smooth Chern connection under the

Bargmann-Fock metric, let

$$g(z) := \nabla_{h_{BF}} f(z) = \frac{\partial f}{\partial z} - \bar{z}f,$$

then it's easy to see the distribution of zeros of $g(z)$ is also invariant under the rotation and translation (by computing the covariance kernel of $g(z)$). Hence, the universal rescaling limit $K_{BF}(u, v) := \lim_{n \rightarrow \infty} K_n(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}})$ is actually a function in the form of $K(|u-v|)$, i.e., it's independent of the position of $z \in M$ and it's a function depending only on the distance of $|u-v|$. But, let the meromorphic derivative

$$g^{mero}(z) := \frac{\partial f}{\partial z},$$

then it's easy to show that the zero set of g^{mero} is only rotation invariant but not translation invariant, and hence, the universal rescaling limit $K_{BF}^{mero}(u, v) := \lim_{n \rightarrow \infty} K_n^{mero}(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}})$ should be a function in the form of $K^{mero}(z, |u-v|)$, i.e., it's a function depending on the position z and the distance of $|u-v|$.

The article is organized as follows. In §2, we will first recall some basic concepts about the positive holomorphic line bundles and Kähler manifolds, then we will define Gaussian random holomorphic sections. In §3, we will derive a Kac-Rice type formula for the two-point correlation of zeros and critical points of Gaussian random holomorphic sections on any Kähler manifold. In §4, we will see that the two-point correlation has a universal rescaling limit since the Bergman kernel does. Then we will derive the estimates (5): we will prove such estimates for Riemann surfaces, then sketch the proof for higher dimensions.

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2. BACKGROUND

In this section, we will review some basic concepts and notations on Gaussian random holomorphic sections of positive holomorphic line bundles over Kähler manifolds.

2.1. Kähler manifolds. Let (M, ω) be a compact Kähler manifold of complex m -dimensional with the Kähler form

$$(6) \quad \omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi,$$

where ϕ is the smooth local Kähler potential in a local coordinate patch $U \subset M$. Let $(L, h) \rightarrow (M, h)$ be a positive holomorphic line bundle such that the curvature of the Hermitian metric h

$$(7) \quad \Theta_h = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log h$$

is a positive $(1, 1)$ form [8]. Let e be a local non-vanishing holomorphic section of L over $U \subset M$ such that locally $L|_U \cong U \times \mathbb{C}$ and the pointwise h -norm of e is $|e|_h = h(e, e)^{1/2}$. Throughout the article, we assume that the line bundle is polarized, i.e.,

$$(8) \quad \Theta_h = \omega \text{ or equivalently } |e|_h^2 = h(e, e) = e^{-\phi}.$$

Thus, $\frac{\omega}{\pi}$ is a de Rham representative of the Chern class $c_1(L)$. Let

$$(9) \quad dV = \frac{\omega^m}{\pi^m m!}$$

be the volume form. We assume that the total volume is

$$\int_M dV = 1.$$

We denote by $H^0(M, L^n)$ the space of global holomorphic sections of the n -th tensor power of L . Locally, we can write the global holomorphic section of L^n as $s_n = f_n e^{\otimes n}$ where f_n is some holomorphic function on U . We denote the dimension of $H^0(M, L^n)$ by d_n . The Hermitian metric h induces a Hermitian metric h^n on L^n as $|e^{\otimes n}|_{h^n} = |e|_h^n$, i.e., $|s_n|_{h^n}^2 = |f_n|^2 h^n(e^{\otimes n}, e^{\otimes n}) = |f_n|^2 e^{-n\phi}$.

Now we can define an inner product on $H^0(M, L^n)$ as the following integration

$$(10) \quad \langle s_{n,1}, s_{n,2} \rangle_{h^n} := \int_M h^n(s_{n,1}, s_{n,2}) dV = \int_M f_{n,1} \overline{f_{n,2}} e^{-n\phi} dV$$

for $s_{n,j} = f_{n,j} e^{\otimes n} \in H^0(M, L^n)$ with $j = 1, 2$.

The Chern connection ∇_{h^n} of the line bundle (L^n, h^n) is the unique connection which is compatible with the Hermitian metrics h^n and the holomorphic structure of complex manifolds [8]. The smooth Chern connection can be decomposed into holomorphic and antiholomorphic parts as

$$(11) \quad \nabla_{h^n} = \nabla'_{h^n} + \nabla''_{h^n},$$

where in the local coordinate, they read

$$(12) \quad \nabla'_{h^n} = d_z + n \partial \log h \text{ and } \nabla''_{h^n} = d_{\bar{z}}.$$

For the polarized line bundle with $h = e^{-\phi}$, the Chern connection is

$$(13) \quad \nabla'_{h^n} s_n = \sum_{i=1}^m \left(\frac{\partial f_n}{\partial z_i} - n \frac{\partial \phi}{\partial z_i} f_n \right) e^{\otimes n} \otimes dz_i \text{ and } \nabla''_{h^n} s_n = \sum_{i=1}^m \frac{\partial f_n}{\partial \bar{z}_i} e^{\otimes n} \otimes d\bar{z}_i$$

for smooth sections $s_n = f_n e^{\otimes n}$ in the local coordinate. For the special case when s_n is a global holomorphic section, we have

$$(14) \quad \nabla_{h^n} s_n = \nabla'_{h^n} s_n.$$

2.2. Kähler normal coordinate. Given a complex m -dimensional Kähler manifold $(L, h) \rightarrow (M, \omega)$, we freeze at a point z_0 as the origin of the coordinate patch and we can choose a Kähler normal coordinate $\{z_j\}$ as well as an adapted frame e_L of the line bundle L around z_0 . It is well-known that in terms of Kähler normal coordinates $\{z_j\}$, the Kähler potential ϕ has the following expansion in the neighborhood of the origin z_0 ,

$$(15) \quad \phi(z, \bar{z}) = \|z\|^2 - \frac{1}{4} \sum R_{j\bar{k}p\bar{q}}(z_0) z_j \bar{z}_{\bar{k}} z_p \bar{z}_{\bar{q}} + O(\|z\|^5).$$

And thus,

$$(16) \quad \phi(z_0) = 0, \partial \phi(z_0) = 0, \partial^2 \phi(z_0) = 0, \partial \bar{\partial} \phi(z_0) = 1, \omega(z_0) = d\ell_z,$$

where $d\ell_z := \frac{i}{2} \sum_{j=1}^m dz_j \wedge d\bar{z}_j$. In general, ϕ contains a pluriharmonic term $f(z) + \overline{f(z)}$, but a change of frame for L eliminates that term up to fourth order. We refer to §3.1 in [4] for more details.

An example on the Kähler normal coordinate and the adapted frame is the affine coordinate for the Fubini-Study metric of the hyperplane line bundle over the complex projective space $(\mathcal{O}(1), h_{FS}) \rightarrow (\mathbb{CP}^1, \omega_{FS})$. The Kähler form on \mathbb{CP}^1 is the Fubini-Study form. In an affine coordinate, the Kähler form and the Kähler potential for the Fubini-Study metric are

$$(17) \quad \omega_{FS} = \frac{\sqrt{-1}}{2} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}, \quad \phi_{FS}(z) = \log(1 + |z|^2).$$

It's easy to check that ϕ_{FS} satisfies (16) and the affine coordinate is actually the Kähler normal coordinate at $z_0 = 0$. We equip $\mathcal{O}(1)$ with its Fubini-Study metric. In fact, we can choose an adapted frame $e(z)$ such that

$$(18) \quad |e(z)|_{h_{FS}}^2 = e^{-\phi} = \frac{1}{1 + |z|^2}.$$

2.3. Bergman kernels. The Bergman kernel is the orthogonal projection from the L^2 -integral sections to the holomorphic sections

$$(19) \quad \Pi_n(z, w) : L^2(M, L^n) \rightarrow H^0(M, L^n)$$

with respect to the inner product (10). It has the following reproducing property

$$(20) \quad \langle s_n(z), \Pi_n(z, w) \rangle_{h^n} = s_n(w),$$

where $s_n \in H^0(M, L^n)$ is a global holomorphic section. Let $\{s_{n,1}, \dots, s_{n,d_n}\}$ be any orthonormal basis of $H^0(M, L^n)$ with respect to the inner product (10), then we have,

$$(21) \quad \Pi_n(z, w) = \sum_{j=1}^{d_n} s_{n,j}(z) \otimes \overline{s_{n,j}(w)}.$$

We write $s_{n,j} = f_{n,j} e^{\otimes n}$ locally, then we can rewrite

$$(22) \quad \Pi_n(z, w) := F_n(z, w) e^{\otimes n}(z) \otimes \overline{e^{\otimes n}(w)}$$

with the local function

$$(23) \quad F_n(z, w) = \sum_{j=1}^{d_n} f_{n,j}(z) \overline{f_{n,j}(w)},$$

where $F_n(z, w)$ is holomorphic in z and anti-holomorphic in w .

The pointwise h^n -norm of the Bergman kernel has the following Tian-Yau-Zelditch C^∞ -expansion on the diagonal [13, 14, 15],

$$(24) \quad |\Pi_n(z, z)|_{h^n} = F_n(z, z) e^{-n\phi} = n^m (1 + a_1(z) n^{-1} + a_2(z) n^{-2} + \dots),$$

where all terms a_j are computable and they are polynomials of curvatures, in particular, a_1 is the scalar curvature of ω .

Take the hyperplane line bundle $\mathcal{O}(1)$ over the complex projective space \mathbb{CP}^1 for example, the global holomorphic sections of $\mathcal{O}(1)$ are linear functions on \mathbb{C}^2 and hence the global holomorphic sections of $L^n = \mathcal{O}(n)$ are homogeneous polynomials of degree n . By choosing Fubini-Study metrics on $(\mathcal{O}(1), h_{FS}) \rightarrow (\mathbb{CP}^1, \omega_{FS})$, an orthonormal basis of $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ under the inner product (10) is given by

$$(25) \quad \left\{ \left(\sqrt{(n+1) \binom{n}{j}} z^j \right) e^{\otimes n} \right\}_{j=0}^n.$$

Thus, the Bergman kernel for the Fubini-Study case is

$$(26) \quad F_n^{FS}(z, w) = (n+1)(1 + z\bar{w})^n.$$

2.4. Gaussian random fields. Let's recall that a complex Gaussian measure on \mathbb{C}^k is a measure of the form

$$(27) \quad d\gamma_\Delta = \frac{e^{-z^* \Delta^{-1} z}}{\pi^k \det \Delta} dV_z,$$

where dV_z denotes Lebesgue measure on \mathbb{C}^k and Δ is a positive definite Hermitian $k \times k$ matrix. The matrix Δ is the covariance matrix.

The inner product (10) induces a complex Gaussian probability measure $d\gamma_{d_n}$ on the space $H^0(M, L^n)$ as,

$$(28) \quad d\gamma_{d_n}(s_n) = \frac{e^{-|a|^2}}{\pi^{d_n}} da, \quad s_n = \sum_{j=1}^{d_n} a_j s_{n,j},$$

where $\{s_{n,1}, \dots, s_{n,d_n}\}$ is an orthonormal basis for $H^0(M, L^n)$ and $\{a_1, \dots, a_{d_n}\}$ are i.i.d. standard complex Gaussian random variables with mean 0 and variance 1.

Thus, by discarding the local frame, the covariance kernel of the Gaussian random section s_n is given by

$$(29) \quad \text{Cov}(s_n(z), s_n(w)) = F_n(z, w),$$

i.e., the Bergman kernel.

3. KAC-RICE TYPE FORMULA

In this section, we will derive a Kac-Rice type formula for the global expression of two-point correlation of $K_n^m(z, w)$. The formula may be derived from [1, 3, 4, 5] but we take advantage of some simplifications to speed up the proof. We will only derive the formula on Riemann surfaces, then generalize naturally to higher dimensions.

3.1. Kac-Rice formula. We will prove the following Kac-Rice type formula for the two-point correlation on Riemann surfaces,

Lemma 1. *On Riemann surfaces (M, ω) , the $(2, 2)$ -current of the two-point correlation of zeros and critical points of holomorphic sections s_n with respect to the Gaussian measure $d\gamma_{d_n}$ is*

$$(30) \quad K_n^1(z, w) = \left(\pi^2 \int_{\mathbb{C}^3} p_{z,w}^n(0, 0, \xi_1, \xi_2, \xi_3) |\xi_1|^2 ||\xi_2|^2 - |\xi_3|^2| dV_\xi \right) dV(z) dV(w),$$

where dV_ξ is the Lebesgue measure on \mathbb{C}^3 , $dV = \frac{\omega}{\pi}$ is the volume form on the Riemann surface and $p_{z,w}^n(x, y, \xi_1, \xi_2, \xi_3)$ is the joint density of Gaussian processes $(s_n(z), \nabla'_{h^n} s_n(w), \nabla'_{h^n} s_n(z), \nabla'_{h^n} \nabla'_{h^n} s_n(w), \nabla''_{h^n} \nabla'_{h^n} s_n(w))$ which can be expressed by the Bergman kernel and its Chern derivatives up to order 4.

Proof. The strategy to get this formula is to find the local expression for the two-point correlation under the local coordinate, then we turn it to be global.

We denote the zero set

$$\mathcal{Z} := \{z \in M : s_n(z) = 0\}$$

and the critical point set

$$\mathcal{C} := \{z \in M : \nabla_{h^n} s_n(z) = 0\}.$$

In the local coordinate $U \cong \mathbb{C}$ and a local trivialization of L , we write Gaussian random holomorphic sections as $s_n = f_n e^{\otimes n}$ where f_n is a holomorphic function, we denote locally

$$g_n(z) := \nabla_{h^n} s_n = \left(\frac{\partial f_n}{\partial z} - n \frac{\partial \phi}{\partial z} f_n \right) e^{\otimes n} \otimes dz.$$

Then the set of critical points of s_n is the same as zeros of $g_n = 0$ (recall definition of Chern connection (13)). We denote locally $\mathcal{Z}_U := \{z \in \mathbb{C} : f_n(z) = 0\}$ as zeros in U ; denote $\mathcal{C}_U := \{z \in \mathbb{C} : g_n(z) = 0\}$ as the set of critical points in U .

By definition of the delta function, for any smooth test functions ψ and φ on M , we have locally,

$$\begin{aligned} & \left\langle \sum_{z \in \mathcal{Z}_U} \delta_z \otimes \sum_{w \in \mathcal{C}_U} \delta_w, \psi \otimes \varphi \right\rangle \\ &= \sum_{z: f_n(z)=0} \psi(z) \sum_{w: g_n(w)=0} \varphi(w) \\ &= \int_{\mathbb{C} \times \mathbb{C}} \delta_0(f_n(z)) \delta_0(g_n(w)) \psi(z) \varphi(w) df_n(z) \wedge d\bar{f}_n(z) \wedge dg_n(w) \wedge d\bar{g}_n(w) \\ &= \int_{\mathbb{C} \times \mathbb{C}} \delta_0(f_n) \delta_0(g_n) \psi(z) \varphi(w) \left| \frac{\partial f_n}{\partial z} \right|^2 \left| \frac{\partial g_n}{\partial w} \right|^2 - \left| \frac{\partial g_n}{\partial \bar{w}} \right|^2 \right| d\ell_z d\ell_w, \end{aligned}$$

where we denote $d\ell$ as the Lebesgue measure on \mathbb{C} .

Now we can turn the above integral to be global by the following observations (by discarding the local frame). Let's first recall the decomposition of the Chern connection $\nabla_{h^n} = \nabla'_{h^n} + \nabla''_{h^n}$ with $\nabla'_{h^n} = d_z - n \frac{\partial \phi}{\partial z}$ and $\nabla''_{h^n} = d_{\bar{z}}$. At z_0 , the zero of the holomorphic section s_n where $f_n(z_0) = 0$, we have

$$\nabla'_{h^n} f_n(z_0) = \frac{\partial f_n}{\partial z}(z_0) - n \frac{\partial \phi}{\partial z} f_n(z_0) = \frac{\partial f_n}{\partial z}(z_0).$$

At the critical point w_0 with $g_n(w_0) = 0$, recall the definition $g_n := \nabla_{h^n} f_n = \nabla'_{h^n} f_n$, by taking derivatives on both sides, we have

$$\nabla'_{h^n} \nabla'_{h^n} f_n(w_0) = \nabla'_{h^n} g_n(w_0) = \frac{\partial g_n}{\partial w}(w_0) - n \frac{\partial \phi}{\partial w} g_n(w_0) = \frac{\partial g_n}{\partial w}(w_0)$$

and

$$\nabla''_{h^n} \nabla'_{h^n} f_n(w_0) = \nabla''_{h^n} g_n(w_0) = \frac{\partial g_n}{\partial \bar{w}}(w_0).$$

Hence, the global expression for the above integration is,

$$\begin{aligned} & \int_{M \times M} \delta_0(s_n(z)) \delta_0(\nabla'_{h^n} s_n(w)) \psi(z) \varphi(w) \\ & \quad \times |\nabla'_{h^n} s(z)|^2 \left| |\nabla'_{h^n} \nabla'_{h^n} s_n(w)|^2 - |\nabla''_{h^n} \nabla'_{h^n} s_n(w)|^2 \right| \omega_z \wedge \omega_w. \end{aligned}$$

By taking the expectation on both sides, we have globally,

$$\begin{aligned} & \mathbf{E} \left\langle \sum_{z \in \mathcal{Z}} \delta_z \otimes \sum_{w \in \mathcal{C}} \delta_w, \psi \otimes \varphi \right\rangle \\ &= \int_{M \times M} \psi(z) \varphi(w) \left(\int_{\mathbb{C}^3} p_{z,w}^n(0, 0, \xi_1, \xi_2, \xi_3) |\xi_1|^2 \left| |\xi_2|^2 - |\xi_3|^2 \right| dV_\xi \right) \omega_z \wedge \omega_w, \end{aligned}$$

where dV_ξ is the Lebesgue measure on \mathbb{C}^3 and $p_{z,w}^n(x, y, \xi_1, \xi_2, \xi_3)$ is the joint density of Gaussian processes $(s_n(z), \nabla'_{h^n} s_n(w), \nabla'_{h^n} s_n(z), \nabla'_{h^n} \nabla'_{h^n} s_n(w), \nabla''_{h^n} \nabla'_{h^n} s_n(w))$.

Hence, the two-point correlation in the case of Riemann surfaces is given by

$$\begin{aligned} \mathbf{E} \left(\sum_{z \in \mathcal{Z}} \delta_z \otimes \sum_{w \in \mathcal{C}} \delta_w \right) &:= K_n^1(z, w) \\ &= \left(\int_{\mathbb{C}^3} p_{z,w}^n(0, 0, \xi_1, \xi_2, \xi_3) |\xi_1|^2 ||\xi_2|^2 - |\xi_3|^2| dV_\xi \right) \omega(z) \wedge \omega(w). \end{aligned}$$

The extra factor π^2 in Lemma 1 appears since we define the volume form $dV := \frac{\omega}{\pi}$. For the last statement, note that the covariance kernel of the Gaussian process s_n is the Bergman kernel, i.e., $\mathbf{E}(s_n(z) \overline{s_n(w)}) = F_n(z, w)$ (see (29)), hence the covariance matrix of Gaussian processes $(s_n(z), \nabla'_{h^n} s_n(w), \nabla' s_n(z), \nabla'_{h^n} \nabla'_{h^n} s_n(w), \nabla''_{h^n} \nabla'_{h^n} s_n(w))$ can be expressed by the Bergman kernel and its Chern derivatives up to order 4 (see [1]), this completes the proof of Lemma 1. \square

3.2. Higher dimensions. For higher dimensions, given a smooth section $s_n = f_n e^{\otimes n}$, the Chern connection has the decomposition (see (13)),

$$\nabla'_{h^n} s_n = \sum_{i=1}^m \left(\frac{\partial f_n}{\partial z_i} - n \frac{\partial \phi}{\partial z_i} f_n \right) e^{\otimes n} \otimes dz_i, \quad \nabla''_{h^n} s_n = \sum_{i=1}^m \frac{\partial f_n}{\partial \bar{z}_i} e^{\otimes n} \otimes d\bar{z}_i.$$

We further rewrite ∇'_{h^n} and ∇''_{h^n} as,

$$\nabla'_{h^n} = \sum_{i=1}^m \nabla'_{h^n, i}, \quad \nabla''_{h^n} = \sum_{i=1}^m \nabla''_{h^n, i},$$

where we define

$$\nabla'_{h^n, i} s_n = \left(\frac{\partial f_n}{\partial z_i} - n \frac{\partial \phi}{\partial z_i} f_n \right) e^{\otimes n} \otimes dz_i, \quad \nabla''_{h^n, i} s_n = \frac{\partial f_n}{\partial \bar{z}_i} e^{\otimes n} \otimes d\bar{z}_i.$$

Following the computations in §3.1 (or §2 in [4]), we have,

Lemma 2. *The $(m+1, m+1)$ -current of the two-point correlation of zeros and critical points of Gaussian holomorphic sections on any compact Kähler manifold of complex m -dimensional is*

$$\begin{aligned} K_n^m(z, w) &= \left(\pi^{m+1} \int_{\mathbb{C}^{m^2+2m}} p_{z,w}^n(0, 0, \xi, H_1, H_2) \|\xi\|^2 |\det(H_1^* H_1 - H_2^* H_2)| dV_\xi dV_H \right) \\ &\quad \times \frac{\omega(z)}{\pi} \wedge \frac{\omega^m(w)}{\pi^m m!}, \end{aligned}$$

where dV_ξ and dV_H are Lebesgue measures on $\xi \in \mathbb{C}^m$ and $(H_1, H_2) \in \mathbb{C}^{m(m+1)}$ where H_1 and H_2 are two symmetric $m \times m$ matrices, $\|\xi\|^2$ is the norm square of the vector ξ and $p_{z,w}^n(x, y, \xi, H_1, H_2)$ is the joint density of Gaussian processes $(s_n(z), (\nabla'_{h^n, i} s_n(w))_{i=1}^m, (\nabla'_{h^n, i} s_n(z))_{i=1}^m, (\nabla'_{h^n, i} \nabla'_{h^n, j} s_n(w))_{i,j}, (\nabla''_{h^n, i} \nabla'_{h^n, j} s_n(w))_{i,j})$ with $1 \leq i \leq j \leq m$.

4. UNIVERSALITY AND SCALING

It can be tell from Lemmas 1 and 2 that the two-point correlation is expressed by the Bergman kernel and its Chern derivatives, hence, the rescaling limits of two-point correlation depend only on the rescaling limits of Bergman kernel and its Chern derivatives. Our plan is the following. In subsection §4.1, we will describe the Bergman kernel for the Bargmann-Fock space. This model case provides the universal rescaling limit for Bergman kernels on any Kähler manifold. Actually, the universal rescaling limit is the Bergman kernel of the Bargmann-Fock space of level 1. In §4.2, we will prove Theorem 1 for Riemann surfaces. In §4.3, we will sketch the proof for the higher dimensions.

4.1. Bargmann-Fock space. The Bargmann-Fock space is the space of entire functions on \mathbb{C}^m which are L^2 -integral with respect to the Bargmann-Fock metric. To be more precise, let's take the trivial line bundle $(L := \mathbb{C} \times \mathbb{C}^m, h_{BF}(z))$ over $(\mathbb{C}^m, \pi^{-m} dV_z)$ with the Hermitian Bargmann-Fock metric $h_{BF}(z) := e^{-\|z\|^2}$ and the Lebesgue measure dV_z on \mathbb{C}^m , here we denote $\|z\|^2 = |z_1|^2 + \cdots + |z_m|^2$. The line bundle is trivial, we may use the frame $e_U = e_{\mathbb{C}} = 1$. By (13), the Chern connection in this case is given by

$$(31) \quad \nabla_{h_{BF}} = \nabla'_{h_{BF}} + \nabla''_{h_{BF}} \text{ with } \nabla'_{h_{BF}} = \sum (d_{z_j} - \bar{z}_j) \text{ and } \nabla''_{h_{BF}} = \sum d_{\bar{z}_j}.$$

We raise the power of the line bundle to $L^{\otimes n}$ and define the Bargmann-Fock space $\mathcal{H}(\mathbb{C}^m, \pi^{-m} e^{-n\|z\|^2} dV_z)$ of level n to be the space of L^2 -entire functions with respect to the inner product (recall (10))

$$(32) \quad \langle f, g \rangle_{h_{BF}^n} = \int_{\mathbb{C}^m} f \bar{g} e^{-n\|z\|^2} \pi^{-m} dV_z.$$

The Bargmann-Fock space is a Hilbert space and the orthonormal basis is given by monomials

$$(33) \quad \left\{ \frac{z^\alpha}{\sqrt{\frac{\alpha!}{n^{m+|\alpha|}}}}, \alpha \in \mathbb{Z}_+^m \right\},$$

where we denote $z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_m|$.

Then the Bergman kernel off the diagonal for the Bargmann-Fock space of level n is (recall (23)),

$$(34) \quad F_n^{BF}(z, w) = \sum_{\alpha \in \mathbb{Z}_+^m} \frac{z^\alpha \bar{w}^\alpha}{\frac{\alpha!}{n^{m+|\alpha|}}} = n^m \sum_{\alpha \in \mathbb{Z}_+^m} \frac{n^{|\alpha|} z^\alpha \bar{w}^\alpha}{\alpha!} = n^m e^{nz \cdot \bar{w}}$$

where $z \cdot \bar{w} = z_1 \bar{w}_1 + \cdots + z_m \bar{w}_m$.

The following asymptotic expansion is proved in [3] which states that the Bergman kernel admits a universal rescaling limit on any Kähler manifold. Let $z_0 \in M$ and choose Kähler normal coordinates in a neighborhood of z_0 and adapted frame e_L , the Bergman kernel admits the full expansion,

$$(35) \quad \begin{aligned} n^{-m} F_n(z_0 + \frac{u}{\sqrt{n}}, z_0 + \frac{v}{\sqrt{n}}) &= e^{u \cdot \bar{v}} + O(n^{-1/2}) \\ &= F_1^{BF}(u, v) + O(n^{-1/2}), \end{aligned}$$

where $F_1^{BF}(u, v)$ is the Bergman kernel for the Bargmann-Fock space of level 1.

The proof of this asymptotic expansion is based on Boutet de Monvel-Sjostrand parametrix construction and the stationary phase method. As a remark, the rescaling limits of Chern derivatives of the Bergman kernel are also universal by taking Chern derivatives on both sides of the above full expansion [3].

An example to illustrate this is the Bergman kernel for the hyperplane line bundle over complex projective space. Recall (26) of Bergman kernel for the Fubini-Study metric $(\mathcal{O}(n), h_{FS}) \rightarrow (\mathbb{CP}^1, \omega_{FS})$, by choosing the Kähler normal coordinate at z_0 , the rescaling limit of the Bergman kernel satisfies the following pointwise limit,

$$(36) \quad \lim_{n \rightarrow \infty} n^{-1} F_n^{FS}(z_0 + \frac{u}{\sqrt{n}}, z_0 + \frac{v}{\sqrt{n}}) = \lim_{n \rightarrow \infty} (1 + \frac{u \cdot \bar{v}}{n})^n = e^{u\bar{v}}.$$

4.2. Proof of Theorem 1 for Riemann surfaces. Let's first derive the universal rescaling limit of the two-point correlation between zeros and critical points on Riemann surfaces. As proved in Lemma 1, it's equivalent to derive the universal rescaling limit of the joint density. This is workable since the joint density is expressed by the Bergman kernel and the Bergman kernel has the universal rescaling limit; furthermore, the limit is achieved by the Bargmann-Fock space of level 1. Hence, following the main idea in [3], to prove the main result for Riemann surfaces, it's enough to consider the following Gaussian analytic functions,

$$(37) \quad f(z) = \sum_{j=0}^{\infty} \frac{a_j}{\sqrt{j!}} z^j,$$

where a_j are i.i.d. standard complex Gaussian random variables with mean 0 and variance 1 and

$$(38) \quad \left\{ \frac{z^j}{\sqrt{j!}} \right\}_{j=0}^{\infty}$$

is an orthonormal basis of the Bargmann-Fock space $\mathcal{H}(\mathbb{C}, \pi^{-1} e^{-|z|^2} dV_z)$ (recall (33)). The two-point correlation between zeros and critical points (defined by Chern connection (31)) of $f(z)$ is the rescaling limit of two-point correlation of Gaussian random holomorphic sections on any Kähler manifold because the covariance kernel of $f(z)$ is

$$(39) \quad \text{cov}(f(z), f(w)) = F_1^{BF}(z, w) = e^{z\bar{w}},$$

i.e., the Bergman kernel of the Bargmann-Fock space of level 1.

Because of the universality of Bergman kernels, we have the following

Lemma 3. *On Riemann surfaces, the $(2,2)$ -current of the two-point correlation of Gaussian random holomorphic sections admits the following pointwise universal limit*

$$\lim_{n \rightarrow \infty} K_n^1(z_0 + \frac{u}{\sqrt{n}}, z_0 + \frac{v}{\sqrt{n}}) = K_{BF}^1(u, v),$$

where $K_{BF}^1(u, v)$ is the two-point correlation between zeros and critical points of the Gaussian random analytic function $f(z)$ defined in (37).

We refer to [3] for more details of this lemma. This lemma completes the first part of our main Theorem 1 for Riemann surfaces. In the followings, let's derive the formula for $K_{BF}^1(u, v)$ and estimate $K_{BF}^1(u, v)$ as $|u - v|$ tends to 0 and ∞ .

4.2.1. *Covariance matrix.* By Lemma 1, the two-point correlation between zeros and critical points of the Gaussian random analytic function $f(z)$ is

$$(40) \quad K_{BF}^1(u, v) = \left(\pi^2 \int_{\mathbb{C}^3} p_{u,v}(0, 0, \xi_1, \xi_2, \xi_3) |\xi_1|^2 \left| |\xi_2|^2 - |\xi_3|^2 \right| dV_\xi \right) \frac{d\ell_u}{\pi} \wedge \frac{d\ell_v}{\pi},$$

where $p_{z,w}(0, 0, \xi_1, \xi_2, \xi_3)$ is the joint density of Gaussian processes $(f(u), \nabla'_{h_{BF}} f(v), \nabla'_{h_{BF}} f(u), \nabla'_{h_{BF}} \nabla'_{h_{BF}} f(v), \nabla''_{h_{BF}} \nabla'_{h_{BF}} f(v))$ and we denote $d\ell_z$ as the Lebesgue measure on \mathbb{C} .

By definition of the Chern connection for the Bargmann-Fock metric (31), we have, $\nabla'_{h_{BF}} f(z) = \frac{\partial f}{\partial z} - \bar{z}f$, $\nabla'_{h_{BF}} \nabla'_{h_{BF}} f(z) = \frac{\partial^2 f}{\partial z^2} - 2\bar{z} \frac{\partial f}{\partial z} + \bar{z}^2 f$ and $\nabla''_{h_{BF}} \nabla'_{h_{BF}} f = -f$. For such Gaussian processes with covariance kernel $\mathbf{E}(f(u)f(v)) = e^{u \cdot \bar{v}}$, the covariance matrix is given by [1],

$$(41) \quad \Delta = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}_{5 \times 5},$$

where

$$A = \begin{pmatrix} e^{|u|^2} & (u-v)e^{u\bar{v}} \\ (\bar{u}-\bar{v})e^{v\bar{u}} & e^{|v|^2} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & (u-v)^2 e^{u\bar{v}} & -e^{u\bar{v}} \\ (1 + \bar{u}v + \bar{v}u - |u|^2 - |v|^2)e^{v\bar{u}} & 0 & 0 \end{pmatrix}$$

and

$C =$

$$\begin{pmatrix} e^{|u|^2} & (u-v)(\bar{u}v + \bar{v}u + 2 - |v|^2 - |u|^2)e^{u\bar{v}} & (\bar{u}-\bar{v})e^{u\bar{v}} \\ (\bar{u}-\bar{v})(\bar{u}v + \bar{v}u + 2 - |v|^2 - |u|^2)e^{\bar{u}v} & 2e^{|v|^2} & 0 \\ (u-v)e^{\bar{u}v} & 0 & e^{|v|^2} \end{pmatrix}.$$

Given the covariance matrix, by elementary matrix computations, the joint density in Lemma 1 can be further simplified as [1],

$$(42) \quad p_{u,v}(0, \xi) = \frac{1}{\pi^5} \frac{1}{\det A \det \Lambda} \exp \{ -\xi^* \Lambda^{-1} \xi \},$$

where

$$(43) \quad \Lambda = C - B^* A^{-1} B$$

is a positive symmetric matrix.

We have the following observations to simplify our computations. Since the Bergman kernel for the Bargmann-Fock space is invariant with respect to unitary transformations and equivariant with respect to translations, hence zeros of Gaussian analytic functions $f(z)$ are also invariant with respect to the group of isometric translations, i.e., unitary transformations and translations of \mathbb{C} [12]. By computing the covariance kernel of $\nabla'_{h_{BF}} f$, we can prove that critical points are also rotation and translation invariant. Hence, the two-point correlation of the Gaussian analytic function is a function depending only on the distance $r := |u - v|$. Without loss of generality, we take $u = r$ and $v = 0$, then,

$$A = \begin{pmatrix} e^{r^2} & r \\ r & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & r^2 & -1 \\ 1 - r^2 & 0 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} e^{r^2} & 2r - r^3 & r \\ 2r - r^3 & 2 & 0 \\ r & 0 & 1 \end{pmatrix}.$$

4.2.2. *Short range behavior.* Let's first derive the short range behavior of the two-point correlation of Gaussian analytic functions as $r \rightarrow 0$. It's easy to get

$$\lim_{r \rightarrow 0} \det A = 1 \quad \text{and} \quad \lim_{r \rightarrow 0} \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If we combine (42), we can further rewrite the $(2, 2)$ -current (40) as,

$$(44) \quad K_{BF}^1(u, v) := \tilde{K}_{BF}^1(u, v) \frac{d\ell_u}{\pi} \wedge \frac{d\ell_v}{\pi}$$

where we denote

$$(45) \quad \tilde{K}_{BF}^1(u, v) = \frac{1}{\pi^3 \det A} \int_{\mathbb{C}^3} \frac{e^{-\xi^* \Lambda^{-1} \xi}}{\det \Lambda} (\xi^* P \xi) |\xi^* Q \xi| dV_\xi.$$

as the two-point correlation function.

Now we change variable $\xi \rightarrow \Lambda^{-\frac{1}{2}} \xi$ to get,

$$\tilde{K}_{BF}^1(u, v) = \frac{1}{\pi^3 \det A} \int_{\mathbb{C}^3} e^{-\|\xi\|^2} (\xi^* \Lambda^{\frac{1}{2}} P \Lambda^{\frac{1}{2}} \xi) \left| \xi^* \Lambda^{\frac{1}{2}} Q \Lambda^{\frac{1}{2}} \xi \right| dV_\xi.$$

We observe that $\Lambda^{\frac{1}{2}} P \Lambda^{\frac{1}{2}}$ can be uniformly bounded for r small enough, thus we can change the order of the limit $r \rightarrow 0$ and the integration of ξ . It's easy to see

$$\lim_{r \rightarrow 0} \Lambda^{\frac{1}{2}} P \Lambda^{\frac{1}{2}} = 0,$$

hence,

$$\lim_{u \rightarrow v} \tilde{K}_{BF}^1(u, v) = 0.$$

This proves that there is a 'repulsion' between zeros and critical points of Gaussian analytic functions.

4.2.3. *Long range behavior.* Now we study the long range behavior of the two-point correlation of Gaussian analytic functions as $r \rightarrow \infty$.

As $r \rightarrow \infty$, we can derive the following estimate,

$$\Lambda = \begin{pmatrix} e^{r^2} - (r^2 - 1)^2 & 2r - r^3 & r \\ 2r - r^3 & 2 & 0 \\ r & 0 & 1 \end{pmatrix} + O(r^{-\infty}).$$

Hence, the square root of Λ has the following estimate as $r \rightarrow \infty$,

$$\Lambda^{\frac{1}{2}} = \begin{pmatrix} e^{r^2/2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(r^{-\infty}).$$

Thus, up to $O(r^{-\infty})$ which is negligible, we have,

$$\tilde{K}_{BF}^1(u, v) = \frac{e^{r^2}}{\pi^3 \det A} \int_{\mathbb{C}^3} e^{-\|\xi\|^2} |\xi_1|^2 |2|\xi_2|^2 - |\xi_1|^2| dV_\xi + O(r^{-\infty}).$$

Note $\det A = e^{r^2} - r^2$, thus as $r \rightarrow \infty$,

$$\frac{e^{r^2}}{\det A} = 1 + O(r^{-\infty}).$$

Hence,

$$\begin{aligned} \tilde{K}_{BF}^1(u, v) &= \frac{1}{\pi^3} \int_{\mathbb{C}^3} e^{-\|\xi\|^2} |\xi_1|^2 |2|\xi_2|^2 - |\xi_1|^2| dV_\xi + O(r^{-\infty}) \\ &= \frac{5}{3} + O(r^{-\infty}). \end{aligned}$$

This verifies that there is no correlation between zeros and critical points for Gaussian analytic functions for the long range, roughly speaking, zeros and critical points behave independently if they are far apart. Hence, we complete our main Theorem 1 for Riemann surfaces.

4.3. **Higher dimensions.** Now let's sketch the proof of Theorem 1 for higher dimensional Kähler manifolds. Again, it's enough to consider the following Gaussian analytic functions on \mathbb{C}^m

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^m} a_\alpha \frac{z^\alpha}{\sqrt{\alpha!}}$$

where a_α are standard complex Gaussian random variables so that the covariance kernel

$$\text{Cov}(f(z), f(w)) = e^{z \cdot \bar{w}}.$$

We apply Lemma 2 to \mathbb{C}^m with the Bargmann-Fock metric, the two-point correlation between zeros and critical points for $f(z)$ is,

$$K_{BF}^m(u, v) := \tilde{K}_{BF}^m(u, v) \frac{d\ell_u}{\pi} \wedge \frac{(d\ell_v)^m}{\pi^m m!},$$

where we denote $d\ell_z := \frac{i}{2} \sum_{j=1}^m dz_j \wedge d\bar{z}_j$ such that $\frac{(d\ell_z)^m}{m!}$ is the Lebesgue measure on \mathbb{C}^m and we denote

$$\begin{aligned} \tilde{K}_{BF}^m(u, v) &= \pi^{m+1} \int_{\mathbb{C}^{m^2+2m}} p_{u,v}^n(0, 0, \xi, H_1, H_2) \|\xi\|^2 |\det(H_1^* H_1 - H_2^* H_2)| \\ &\quad \times dV_\xi dV_{H_1} dV_{H_2} \end{aligned}$$

as the two-point correlation function. Here, $p_{u,v}^n(x, y, \xi, H_1, H_2)$ is the joint density of Gaussian processes $(f(u), \nabla'_{h_{BF},i} f(v), \nabla'_{h_{BF},i} f(u), \nabla'_{h_{BF},i} \nabla'_{h_{BF},j} f(v), \nabla''_{h_{BF},i} \nabla'_{h_{BF},j} f(v))$ with $1 \leq i \leq j \leq m$.

By definition of the Chern connection (31), we have $\nabla'_{h_{BF},i} f = \frac{\partial f}{\partial z_i} - \bar{z}_i f$, $\nabla'_{h_{BF},j} \nabla'_{h_{BF},i} f = \frac{\partial^2 f}{\partial z_i \partial z_j} - \bar{z}_i \frac{\partial f}{\partial z_j} - \bar{z}_j \frac{\partial f}{\partial z_i} + \bar{z}_i \bar{z}_j f$ and $\nabla''_{h_{BF},j} \nabla'_{h_{BF},i} f = -\delta_i^j f$. Both zeros and critical points of Gaussian analytic functions $f(z)$ are rotation and translation invariant, hence, two-point correlation is again a function depending only on the distance $r := |u - v|$. Without loss of generality, we take $u = (r, 0, \dots, 0)$ and $v = (0, \dots, 0)$. The Gaussian processes are further simplified to be

$$\left(f(u), \frac{\partial f(v)}{\partial v_i}, \frac{\partial f(u)}{\partial u_i} - \delta_1^i r f(u), \frac{\partial^2 f}{\partial v_i \partial v_j}, -\delta_i^j f(v) \right)$$

evaluated at the point $u = (r, 0, \dots, 0)$ and $v = (0, \dots, 0)$ where $1 \leq i \leq j \leq m$.

Note that the last element is $-\delta_i^j f(v)$, this implies that the dimension of the above Gaussian processes can be reduced when $u = (r, 0, \dots, 0)$ and $v = (0, \dots, 0)$. Hence, the two-point correlation can be further simplified to be

$$(46) \quad \begin{aligned} \tilde{K}_{BF}^m(u, v) &= \pi^{m+1} \int_{\mathbb{C}^{\frac{(m+1)(m+2)}{2}}} p_{u,v}^n(0, 0, \xi, H_1, \eta) \|\xi\|^2 |\det(H_1^* H_1 - |\eta|^2 I)| \\ &\quad \times dV_\xi dV_{H_1} dV_\eta \end{aligned}$$

evaluated at $u = (r, 0, \dots, 0)$ and $v = (0, \dots, 0)$, where dV_η is the Lebesgue measure on \mathbb{C} . Here, $p_{u,v}^n(x, y, \xi, H_1, \eta)$ is the joint density of Gaussian processes

$$\left(f(u), \frac{\partial f(v)}{\partial v_i}, \frac{\partial f(u)}{\partial u_i} - \delta_1^i r f(u), \frac{\partial^2 f}{\partial v_i \partial v_j}, -f(v) \right)$$

evaluated at the point $u = (r, 0, \dots, 0)$ and $v = (0, \dots, 0)$.

To compute the covariance matrix, following identities (42)(43), we have,

$$\begin{aligned} A &= \begin{pmatrix} \mathbf{E} f(u) \overline{f(u)} & \mathbf{E} f(u) \overline{\frac{\partial f(v)}{\partial v_i}} \\ \mathbf{E} \overline{f(u)} \frac{\partial f(v)}{\partial v_i} & \mathbf{E} \frac{\partial f(v)}{\partial v_i} \overline{\frac{\partial f(v)}{\partial v_j}} \end{pmatrix} \Big|_{u=(r,0,\dots,0), v=(0,\dots,0)} \\ &= \begin{pmatrix} e^{r^2} & r & 0 & \dots & 0 \\ r & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{(m+1) \times (m+1)}. \end{aligned}$$

Following the same computations, we have,

$$B = \left(\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & r^2 & 0 & \cdots & 0 & -1 \\ 1-r^2 & 0 & \cdots & & & & & & \\ 0 & 1 & \cdots & & & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 1 & & & & & \end{array} \right)_{(m+1) \times \frac{(m+1)(m+2)}{2}}$$

and C is a symmetric $\frac{(m+1)(m+2)}{2} \times \frac{(m+1)(m+2)}{2}$ matrix sketched as,

$$C = \left(\begin{array}{cccc|cccc} e^{r^2} & 0 & \cdots & 0 & 2r-r^3 & 0 & \cdots & 0 & r \\ 0 & e^{r^2} & 0 & \cdots & & r & & \cdots & 0 \\ \vdots & & e^{r^2} & 0 & \cdots & & r & \cdots & 0 \\ & & & \ddots & & & & \vdots & \vdots \\ 2r-r^3 & & & & 2 & & & 0 & 0 \\ \vdots & & & & & 1 & & 0 & 0 \\ & & & & & & \ddots & \vdots & \vdots \\ r & \cdots & & & \cdots & & & 2 & 0 \\ & & & & & & & \cdots & 1 \end{array} \right)$$

The diagonal elements of C is $\{\underbrace{e^{r^2}, \dots, e^{r^2}}_m, \underbrace{2, 1, \dots, 1}_m, \underbrace{2, 1, \dots, 1}_{m-1}, \dots, \underbrace{2, 1, 2, 1}_2\}$.

For the off diagonal element, it's either 0 or $2r-r^3$ or r . In fact, we will see that the diagonal elements especially the first m diagonal elements in C are crucial in the following computations.

For the short range as $r \rightarrow 0$, the matrix A tends to the identity matrix, B tends to a matrix with $(0, \dots, -1)$ as the first row and a $m \times m$ identity matrix in the lower-triangle and 0 for the rest, and C tends to a diagonal matrix $\text{diag}\{\underbrace{1, 1, \dots, 1}_m, 2, 1, \dots, 2\}$. Hence, as $r \rightarrow 0$, $\Lambda = C - B^* A^{-1} B$ tends to a diagonal matrix $\text{diag}\{\underbrace{0, 0, \dots, 0}_m, 2, \dots\}$ where the first m elements are 0. Hence, the

Gaussian density at least degenerates to $\delta_{\xi=0}$ as $r \rightarrow 0$ for $\xi \in \mathbb{C}^m$, which implies that the integration (46) must tend to 0.

For the long range as $r \rightarrow \infty$, following the same argument as in §4.2.3, we change variables $\xi \rightarrow e^{r^2/2}\xi$, then up to a negligible term $O(r^{-\infty})$, the limit as $r \rightarrow \infty$ is the constant

$$(47) \quad c_m = \frac{\pi^{m+1}}{\det \tilde{\Lambda}} \int_{\mathbb{C}^{\frac{(m+1)(m+2)}{2}}} \exp \left\{ -(\xi, H_1, \eta) \tilde{\Lambda}^{-1} \begin{pmatrix} \xi \\ H_1 \\ \eta \end{pmatrix} \right\} \|\xi\|^2 \\ \times |\det(H_1^* H_1 - |\eta|^2 I)| dV_{\xi} dV_{H_1} dV_{\eta}$$

where $\tilde{\Lambda}$ is the diagonal matrix $\text{diag}\{\underbrace{1, \dots, 1}_m, \underbrace{2, 1, \dots, 1}_m, \underbrace{2, 1, \dots, 1}_{m-1}, \dots, \underbrace{2, 1, 2, 1}_2\}$.

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